

On the multiplicity of solutions of a system of algebraic equations

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We obtain upper bounds for the multiplicity of an isolated solution of a system of equations $f_1 = \dots = f_M = 0$ in M variables, where the set of polynomials (f_1, \dots, f_M) is a tuple of general position in a subvariety of a given codimension which does not exceed M , in the space of tuples of polynomials. It is proved that for $M \rightarrow \infty$ that multiplicity grows not faster than $\sqrt{M} \exp[\omega\sqrt{M}]$, where $\omega > 0$ is a certain constant.

Bibliography: 3 titles.

Introduction

In the present paper, the following problem is considered. Let

$$\begin{cases} f_1(z_1, \dots, z_M) = 0 \\ \dots \\ f_M(z_1, \dots, z_M) = 0 \end{cases}$$

be a system of polynomial equations of degree $d \geq 2$, which has the origin $o = (0, \dots, 0) \in \mathbb{C}^M$ as an isolated solution. For a given $m \geq 1$ one needs to estimate the codimension of the set of such tuples (f_1, \dots, f_M) , that

$$\dim \mathcal{O}_{o, \mathbb{C}^M} / (f_1, \dots, f_M) \geq m,$$

in the space of all tuples of polynomials of degree d with no free term. Informally speaking, how many independent conditions on the coefficients of the polynomials f_1, \dots, f_M are imposed if it is required that the multiplicity of the given solution is no smaller than m ? Problems of that type emerge in the theory of birational rigidity (see [1, Proposition 3.3]). As another application, we point out the problem of description of possible singularities of the variety of lines on a generic Fano variety $V \subset \mathbb{P}^N$ in a given family. However, this problem is interesting by itself, too. The problem described above can be formulated in another way: for a given codimension $a \geq 1$ to estimate the maximal possible multiplicity for a *generic* tuple of equations $(f_1, \dots, f_M) \in B$ in a *given* subvariety B of codimension a . Thus we are looking for the maximum over all subvarieties B , in each of which a tuple of general position is taken. Of course, this problem makes sense only provided that the set of such tuples (f_1, \dots, f_M) , that the system $f_1 = \dots = f_M = 0$ has a set of solutions of positive

dimension, containing the point o , is of codimension not less than $a + 1$. This is true, if $a \leq M$.

In [1,§3] a simple example (the idea of which is actively used in this paper) was constructed, which shows that for $M \gg 0$ the maximal multiplicity of an isolated solution in codimension $a = M$ grows not slower than

$$2^{\sqrt{M}}.$$

In the present paper for this value we obtain the upper bound

$$\sqrt{M}e^{\omega\sqrt{M}},$$

where $\omega > 0$ is a certain concrete real number. To do this, we generalize the problem above for systems of $i \leq M$ polynomial equations, which makes it possible to construct an inductive procedure of estimating the maximal intersection multiplicity for a given codimension of the set of equations.

Let us explain the main difficulty in solving the problem above. Let Y_i , $i = 1, \dots, M$, be the algebraic cycle of the scheme-theoretic intersection

$$(\{f_1 = 0\} \circ \dots \circ \{f_i = 0\})$$

in a neighborhood of the point o . This is an effective cycle of codimension i . Set $m_i = \text{mult}_o Y_i$. It seems natural to consider the whole sequence of multiplicities (m_1, \dots, m_M) , estimating the codimension of the space of polynomials f_{i+1} in terms of the jump of the multiplicity from m_i to m_{i+1} (this very approach was realized in [1,§3]). However, in our problem this approach does not work.

Let $\widetilde{\mathbb{C}^M} \rightarrow \mathbb{C}^M$ be the blow up of the point o , $E \cong \mathbb{P}^{M-1}$ the exceptional divisor, \widetilde{Y}_i the strict transform of Y_i , $(\widetilde{Y}_i \circ E) = \sum c_j R_j$ the algebraic projectivized tangent cone. According to the intersection theory [2], the multiplicity of the scheme-theoretic intersection of the cycle Y_i and the divisor $D_{i+1} = \{f_{i+1} = 0\}$ at the point o is given by the formula

$$m_{i+1} = m_i \text{mult}_o D_{i+1} + \sum_{R_{jk}} d_{jk} \left(\text{mult}_{R_{jk}} \widetilde{Y} \right) \left(\text{mult}_{R_{jk}} \widetilde{D_{i+1}} \right),$$

where the sum is taken over some finite set of irreducible subvarieties of codimension $(i+1)$, including infinitely near ones, R_{jk} covers R_j with the multiplicity d_{jk} . Taking into account that $M \gg 0$, for i close to M the structure of the singularity of the cycle Y_i at the point o can be arbitrary, that is, it can not be explicitly described. Therefore, it is impossible to estimate, how many independent conditions on the polynomial f_{i+1} for f_1, \dots, f_i fixed are imposed by the bounds for the multiplicities $\text{mult}_{R_{jk}} \widetilde{D_{i+1}}$. The only and obvious conclusion, which can be derived from the formula for m_{i+1} , given above, is that the condition $m_{i+1} \geq c$ for a fixed cycle Y_i defines a closed subset in the space of polynomials f_{i+1} , which is a union of a finite number of linear subspaces. Indeed, the condition $\text{mult}_{R_{jk}} \widetilde{D_{i+1}} \geq \gamma$ is a linear one.

By what was said above, in order to get an effective bound for the maximal intersection multiplicity in codimension $a \geq 1$ one needs a different approach, which is developed in the present paper. The main idea is to estimate the maximal multiplicity for i polynomials via the maximal multiplicity for $(i - 1)$ polynomials with an appropriate correction of the codimension. The estimates, obtained by means of this inductive method, seem to be close to the optimal ones.

The paper is organized in the following way. In §1 we develop an inductive procedure of estimating the multiplicity. Using it, in §2 we derive an absolute estimate of the intersection multiplicity and, as a corollary, the main asymptotic result of this paper. In §3, following [3], we briefly remind the method of estimating the codimension of the set of tuples (f_1, \dots, f_i) , defining sets of an “incorrect” codimension $\leq i - 1$.

To conclude, we note that the problem, considered in this paper, can be set up and solved by the same method for an arbitrary very ample class H on an algebraic variety V at a point $o \in V$.

§1. The inductive method of estimating the multiplicity

In this section we develop an inductive procedure of estimating the maximal multiplicity in a given codimension. In the beginning of the section we consider equations of arbitrary degree $d \geq 2$, later we restrict ourselves by quadratic polynomials ($d = 2$). For a codimension, not exceeding M , this does not change the result (see Remark 1.4).

1.1. Set up of the problem. Fix the complex coordinate space $\mathbb{C}_{(z_1, \dots, z_M)}^M$, $M \geq 1$. By the symbol $\mathcal{P}_{d,M}$ we denote the space of *homogeneous* polynomials of degree $d \geq 1$ in the variables z_* , by the symbol $\mathcal{P}_{\leq d,M}$ we denote the space of polynomials of degree $\leq d$ *with no free term* in the variables z_* . On each of these spaces there is a natural action of the matrix group $GL_M(\mathbb{C})$. Set

$$\mathcal{P}_{\leq d,M}^i = \underbrace{\mathcal{P}_{\leq d,M} \times \dots \times \mathcal{P}_{\leq d,M}}_i$$

to be the space of tuples (f_1, \dots, f_i) . By the symbol

$$Z(f_1, \dots, f_i)$$

we denote the subscheme $\{f_1 = \dots = f_i = 0\}$, which we will study in a neighborhood of the point $o = (0, \dots, 0)$, that is, in fact, the subject of our study is the local ring

$$\mathcal{O}_{o, Z(f_*)} = \mathcal{O}_{o, \mathbb{C}^M} / (f_1, \dots, f_i).$$

Denote the map

$$\mu: \mathcal{P}_{\leq d,M}^i \rightarrow \mathbb{Z}_+ \cup \{\infty\},$$

setting $\mu(f_1, \dots, f_i) = \infty$, if $\text{codim}_o Z(f_1, \dots, f_i) \leq i - 1$ (the symbol codim_o stands for the codimension in a neighborhood of the point o), and

$$\mu(f_1, \dots, f_i) = \text{mult}_o Z(f_1, \dots, f_i),$$

if $\text{codim}_o Z(f_1, \dots, f_i) = i$. For an arbitrary irreducible subvariety $B \subset \mathcal{P}_{\leq d, M}^i$ set

$$\mu(B) = \min_{(f_1, \dots, f_i) \in B} \{\mu(f_1, \dots, f_i)\} \in \mathbb{Z}_+ \cup \{\infty\}.$$

Therefore, $\mu(B) = \infty$ if and only if for every tuple of polynomials $(f_1, \dots, f_i) \in B$ the complete intersection $Z(f_1, \dots, f_i)$ has in a neighborhood of the point o an “incorrect” codimension $\leq i - 1$. The equality $\mu(B) = m \in \mathbb{Z}_+$ means that for a generic tuple of polynomials $(f_1, \dots, f_i) \in B$ the complete intersection $Z(f_1, \dots, f_i)$ has in a neighborhood of the point o the correct codimension i and its multiplicity at the point o is $m \geq 1$.

Definition 1.1. The maximal intersection multiplicity of a generic tuple of polynomials at the point o in the codimension $a \in \mathbb{Z}_+$ is

$$\mu_i(a) = \max_{B \subset \mathcal{P}_{\leq d, M}^i} \{\mu(B)\} \in \mathbb{Z}_+ \cup \{\infty\},$$

where the maximum is taken over all irreducible subvarieties $B \subset \mathcal{P}_{\leq d, M}^i$ of codimension a .

Definition 1.1 can be re-formulated as follows. The multiplicity $\mu_i(a)$ is ∞ , if and only if the codimension of the closed algebraic set

$$\{(f_1, \dots, f_i) \in \mathcal{P}_{\leq d, M}^i \mid \mu(f_1, \dots, f_i) = \infty\}$$

does not exceed a (and in that case for B we can take any irreducible subvariety of codimension a , contained in that set). Otherwise, the multiplicity $\mu_i(a)$ is the minimal positive integer $m \geq 1$, satisfying the condition: the codimension of the closed algebraic set

$$\{(f_1, \dots, f_i) \in \mathcal{P}_{\leq d, M}^i \mid \mu(f_1, \dots, f_i) \geq m + 1\}$$

is not less than $a + 1$. In other words, for any irreducible subvariety $B \subset \mathcal{P}_{\leq d, M}^i$ of codimension a and a generic tuple $(f_1, \dots, f_i) \in B$ we get

$$\mu(f_1, \dots, f_i) \leq \mu_i(a)$$

and for a certain subvariety B this inequality turns into the equality.

Remark 1.1. Apart from the matrix group $GL_M(\mathbb{C})$, which acts naturally on the space $\mathcal{P}_{\leq d, M}^i$ by linear changes of coordinates, on that space naturally acts the matrix group $GL_i(\mathbb{C})$: with a non-degenerate $(i \times i)$ matrix A we associate the transformation of the tuple of polynomials

$$(f_1, \dots, f_i) \mapsto (f_1, \dots, f_i)A.$$

The multiplicity $\mu(f_1, \dots, f_i)$ is invariant with respect to the action of these two groups. Respectively, the algebraic sets

$$X_{i,M}(m) = \{(f_1, \dots, f_i) \mid \mu(f_1, \dots, f_i) \geq m\} \subset \mathcal{P}_{\leq d, M}^i$$

and their irreducible components are $GL_M(\mathbb{C})$ - and $GL_i(\mathbb{C})$ -invariant. For this reason, the definition of the number $\mu_i(a)$ can be modified in the following way: for any $GL_M(\mathbb{C})$ - and $GL_i(\mathbb{C})$ -invariant subvariety $B \subset \mathcal{P}_{\leq d, M}^i$ of codimension $\leq a$ we have $\mu(B) \leq \mu_i(a)$, and moreover, for a certain (invariant) B this is an equality. The equivalence of the two definitions of the number $\mu_i(a)$ is obvious: let us consider the closed set $X_{i,M}(\mu_i(a))$. Its codimension in the space $\mathcal{P}_{\leq d, M}^i$ does not exceed a and each of its components is invariant, and moreover, for some component B of codimension $\leq a$ we have $\mu(B) = \mu_i(a)$, which is what we need.

Now let us consider the problem, for which values $a \in \mathbb{Z}_+$ the numbers $\mu_i(a)$ are certainly finite.

Proposition 1.1. *The codimension of the closed set $X_{i,M}(\infty)$ for $i \leq M-1$ is not less than dM , and for $i = M$ not less than $(d-1)M+1$.*

Proof is given in §3.

Corollary 1.1. *For $a \leq M$ we have $\mu_i(a) < \infty$.*

The problem of estimating the numbers $\mu_i(a)$ from above is considered in this paper for those values of a only.

1.2. The invariant ε and reduction to the standard form. For an irreducible subvariety $B \subset \mathcal{P}_{\leq 2, M}^i$ we define the number

$$\varepsilon(B) = i - \text{rk}(df_1(o), \dots, df_i(o)) \in \{0, 1, \dots, i\},$$

where $(f_1, \dots, f_i) \in B$ is a tuple of general position. If the subvariety B is $GL_i(\mathbb{C})$ -invariant, then the equality $\varepsilon(B) = b$ means that in a generic tuple $(f_1, \dots, f_i) \in B$ the first $(i-b)$ linear forms

$$df_1(o), \dots, df_{i-b}(o)$$

are linearly independent, whereas the forms $df_{i-b+j}(o)$ for $j \in \{1, \dots, b\}$ are their linear combinations. For any irreducible subvariety B , satisfying the latter condition, there exists a non-empty Zariski open subset $B^o \subset B$, on which the map of *reducing to the standard form* is well defined:

$$\rho: B^o \rightarrow \mathcal{P}_{\leq 2, M}^{i-b} \times \mathcal{P}_{2, M}^b,$$

$$\rho: (f_1, \dots, f_i) \mapsto (f_1, \dots, f_{i-b}, f_{i-b+1}^+, \dots, f_i^+),$$

where

$$f_{i-b+j}^+ = f_{i-b+j} - \sum_{\alpha=1}^{i-b} \lambda_{j\alpha} f_\alpha,$$

the coefficients $\lambda_{j\alpha}$ are defined by the equalities

$$df_{i-b+j}(o) = \sum_{\alpha=1}^{i-b} \lambda_{j\alpha} df_{\alpha}(o).$$

Therefore, $df_{i-b+j}^+(o) = 0$ and f_{i-b+j}^+ is a homogeneous polynomial of degree 2. The closure of the image $\rho(B^o)$ we denote by the symbol \bar{B} . If the subvariety B is invariant with respect to the action of $GL_i(\mathbb{C})$, then the coefficients $\lambda_{j\alpha}$ take arbitrary values and for that reason

$$\dim B = \dim \bar{B} + b(i - b).$$

Obviously, every fibre of the map $\rho: B^o \mapsto \rho(B^o)$ is $\mathbb{C}^{b(i-b)}$.

The main technical tool for estimating the numbers $\mu_i(a)$ is given by the more sensitive numbers

$$\mu_{i,M}(a, b) = \max \left\{ m \geq 1 \mid \begin{array}{l} \text{the set } X_{i,M} \text{ has an irreducible component} \\ B \text{ of codimension } \leq a \text{ with } \varepsilon(B) = b \end{array} \right\}.$$

Obviously, $\mu_i(a) = \max_b \{\mu_{i,M}(a, b)\}$, where the maximum is taken over all possible values of the number $\varepsilon(B)$ for irreducible subvarieties B of codimension $\leq a$. It is easy to see that the codimension of the subset

$$\{(f_1, \dots, f_i) \mid \text{rk}(df_1(o), \dots, df_i(o)) \leq i - b\}$$

is $b(M + b - i)$, so that the equality $\varepsilon(B) = b$ is only possible if $a \geq b(M + b - i)$. In the sequel, when the notation $\mu_{i,M}(a, b)$ is used, it means automatically that the latter inequality holds. The following obvious fact is true.

Proposition 1.2. *The equality*

$$\mu_{i,M}(a, 0) = 1$$

holds.

Proof. If $\varepsilon(B) = 0$, then for a generic tuple (f_1, \dots, f_i) the differentials $df_1(o), \dots, df_i(o)$ are linearly independent, that is, the set $\{f_1 = \dots = f_i = 0\}$ is a smooth subvariety of codimension i in a neighborhood of the point o , which is what we need. Q.E.D.

Let us find an upper bound for the numbers $\mu_{i,M}(a, b)$ for $b \geq 1$.

1.3. Splitting off the last factor. Let

$$\pi_i: \mathcal{P}_{\leq 2, M}^{i-b} \times \mathcal{P}_{2, M}^b \rightarrow \mathcal{P}_{\leq 2, M}^{i-b} \times \mathcal{P}_{2, M}^{b-1}$$

be the projection along the last direct factor $\mathcal{P}_{2, M}$. For the closed set $\bar{B} \subset \mathcal{P}_{\leq 2, M}^{i-b} \times \mathcal{P}_{2, M}^b$, constructed above, denote by the symbol $[\bar{B}]_{i-1}$ the closure of the set $\pi_i(\bar{B})$. It is easy to see that the following relation holds:

$$\text{codim}(\bar{B} \subset \mathcal{P}_{\leq 2, M}^{i-b} \times \mathcal{P}_{2, M}^b) = \text{codim}(B \subset \mathcal{P}_{\leq 2, M}^i) - (M + b - i)b.$$

Starting from this moment, unless otherwise specified, the codimension is always meant with respect to the natural ambient space; for instance, the last equality writes simply as

$$\text{codim } \bar{B} = \text{codim } B - (M + b - i)b.$$

Sometimes for the convenience of the reader we remind, the codimension with respect to which space is meant.

For a tuple of general position $(f_1, \dots, f_{i-1}) \in [\bar{B}]_{i-1}$ denote by the symbol

$$[\bar{B}]^i = [\bar{B}]^i(f_1, \dots, f_{i-1}) \subset \mathcal{P}_{2,M}$$

the fibre of the projection $\pi_i|_{\bar{B}}: \bar{B} \mapsto [\bar{B}]_{i-1}$. Obviously,

$$\text{codim } \bar{B} = \text{codim}[\bar{B}]_{i-1} + \text{codim}[\bar{B}]^i$$

(recall: the codimension is meant with respect to the natural ambient space, for \bar{B} it is $\mathcal{P}_{\leq 2, M}^{i-b} \times \mathcal{P}_{2, M}^b$, for $[\bar{B}]_{i-1}$ it is the space $\mathcal{P}_{\leq 2, M}^{i-b} \times \mathcal{P}_{2, M}^{b-1}$, for $[\bar{B}]^i$ it is $\mathcal{P}_{2, M}$). Set

$$\gamma_i = \gamma_i(B) = \text{codim}[\bar{B}]^i.$$

Since $\text{codim } B \leq a$, we obtain the estimate

$$\begin{aligned} \text{codim}[\bar{B}]_{i-1} &= \text{codim } B - (M + b - i)b - \gamma_i \leq \\ &\leq a - (M + b - i)b - \gamma_i. \end{aligned}$$

This, in particular, implies that

$$0 \leq \gamma_i \leq a - (M + b - i)b.$$

1.4. The main inductive estimate. The following fact is true.

Theorem 1. *For any i, M, a, b there exist integers $\alpha \in \{0, 1\}$ and $\gamma \in \{0, \dots, a - (M + b - i)b\}$ such that the following inequality holds:*

$$\begin{aligned} \mu_{i,M}(a, b) \leq & \mu_{i-1,M}(a - (M + b - i) - \gamma, b - 1) + \\ & + \mu_{i-1,M-1}(a - (M + b - i) - \alpha(b - 1), b - \alpha). \end{aligned} \quad (1)$$

Remark 1.2. As we will see from the proof of the theorem, the numbers α and γ are determined by the subvariety B , which realizes the multiplicity $\mu_{i,M}(a, b)$. There can be more than one such subvariety; respectively, several inequalities (1) can be satisfied for the number $\mu_{i,M}(a, b)$, with different values of α and γ . Furthermore, the inequalities

$$\mu_{i,M}(a_1, b) \leq \mu_{i,M}(a_2, b) \quad \text{and} \quad \mu_i(a_1) \leq \mu_i(a_2)$$

hold for $a_1 \leq a_2$, which implies that in (1) one can set $\gamma = 0$ and the estimate still holds (possibly becomes weaker).

Proof of Theorem 1. Let us fix a $GL_i(\mathbb{C})$ -invariant irreducible subvariety B , realizing the value $\mu_{i,M}(a, b)$, $\varepsilon(B) = b$. We may assume that B is an irreducible component of the closed set $X_{i,M}(m)$, where $m = \mu_{i,M}(a, b)$. To simplify the formulas, we assume that $\text{codim } B = a$ (if $\text{codim } B < a$, then the estimates below can only become stronger). Fix a linear form $L(z_1, \dots, z_M)$ of general position. In particular, if $(f_1, \dots, f_i) \in B$ is a generic tuple, so that the set $\{f_1 = \dots = f_{i-1} = 0\}$ is of codimension $(i - 1)$ in a neighborhood of the point o , the multiplicity of the effective cycle

$$(\{f_1 = 0\} \circ \dots \circ \{f_{i-1} = 0\})$$

at the point o is equal to the multiplicity of the intersection of that cycle with the hyperplane $\{L = 0\}$ at the point o . Let

$$\Pi_L = \{L(z_*)L_1(z_*) \mid L_1 \in \mathcal{P}_{1,M}\} \subset \mathcal{P}_{2,M}$$

be the linear space of reducible homogeneous quadratic polynomials, divisible by L . Set

$$\mathcal{P}_L = \mathcal{P}_{\leq 2,M}^{i-b} \times \mathcal{P}_{2,M}^{b-1} \times \Pi_L \subset \mathcal{P}_{\leq 2,M}^{i-b} \times \mathcal{P}_{2,M}^b.$$

This is a closed subset. The intersection $\bar{B} \cap \mathcal{P}_L$ is non-empty and of codimension not higher than $\text{codim } \bar{B}$ in \mathcal{P}_L . By the symbol $[\bar{B} \cap \mathcal{P}_L]_{i-1}$ we denote the closure of the set $\pi_i(\bar{B} \cap \mathcal{P}_L)$. As we consider only codimensions $a \leq M$, the equality

$$[\bar{B} \cap \mathcal{P}_L]_{i-1} = [\bar{B}]_{i-1}$$

holds, since for a generic tuple $(f_1, \dots, f_i) \in B$ the intersection of the space Π_L with the fibre $[\bar{B}]^i(f_1, \dots, f_{i-1})$ has a positive dimension. More precisely, the codimension of that intersection in $\Pi_L \cong \mathcal{P}_{1,M}$ does not exceed γ_i .

Remark 1.3. Since we assume that B is an irreducible component of the closed set $X_{i,M}(m)$, the fibre

$$[\bar{B}]^i = \{f_i \in \mathcal{P}_{2,M} \mid \text{mult}_o\{f_1 = \dots = f_i = 0\} \geq m\},$$

$m = \mu_{i,M}(a, b)$, for f_1, \dots, f_{i-1} fixed, is a union of a finite number of linear subspaces of codimension γ_i . Therefore, the closed set

$$\Pi(f_1, \dots, f_{i-1}) = \{L_1 \in \mathcal{P}_{1,M} \mid \text{mult}_o\{f_1 = \dots = f_{i-1} = LL_1 = 0\} \geq m\}$$

for a generic tuple $(f_1, \dots, f_{i-1}) \in [\bar{B}]_{i-1}$ is a union of a finite number of linear subspaces in $\mathcal{P}_{1,M}$, the codimension of each of which in $\mathcal{P}_{1,M}$ does not exceed γ_i .

By what was said, the inequality

$$\begin{aligned} m = \mu_{i,M}(a, b) \leq & \text{mult}_o\{f_1 = \dots = f_{i-1} = L = 0\} + \\ & + \text{mult}_o\{f_1 = \dots = f_{i-1} = L_1 = 0\} \end{aligned}$$

holds. Since L is a form of general position, the first summand in the right hand side is

$$\text{mult}_o\{f_1 = \dots = f_{i-1} = 0\}.$$

Here $(f_1, \dots, f_{i-1}) \in [\bar{B}]_{i-1}$ is a tuple of general position. Now the set $[\bar{B}]_{i-1} \subset \mathcal{P}_{\leq 2, M}^{i-b} \times \mathcal{P}_{2, M}^{b-1}$ can be represented as a result of reducing to the standard form of the closed subset $C \subset \mathcal{P}_{\leq 2, M}^{i-1}$, that is,

$$[\bar{B}]_{i-1} = \bar{C},$$

where C is constructed by the procedure, which is inverse to the procedure of reducing to the standard form: C is the closure of the set of $(i-1)$ -tuples

$$\{(g_1, \dots, g_{i-b}, g_{i-b+1}^+, \dots, g_{i-1}^+)\},$$

where

$$g_{i-b+j}^+ = g_{i-b+j} + \sum_{\alpha=1}^{i-b} \lambda_{j\alpha} g_\alpha,$$

for all $(g_1, \dots, g_{i-1}) \in [\bar{B}]_{i-1}$ and $\lambda_{j\alpha} \in \mathbb{C}$. From this, it follows that

$$\dim C = \dim[\bar{B}]_{i-1} + (b-1)(i-b),$$

so that

$$\begin{aligned} \text{codim } C &= \text{codim}[\bar{B}]_{i-1} + (b-1)(M+b-i) = \\ &= \text{codim } B - (M+b-i) - \gamma_i. \end{aligned}$$

(Recall, that each of the three codimensions is taken with respect of the *corresponding* ambient space; for instance, for C it is $\mathcal{P}_{\leq 2, M}^{i-1}$). Since, obviously, $\varepsilon(C) = b-1$, we obtain that

$$\text{mult}_o\{f_1 = \dots = f_{i-1} = 0\} \leq \mu_{i-1, M}(a - (M+b-i) - \gamma_i, b-1).$$

This gives us the first half of the right hand side of the inequality of Theorem 1.

1.5. The multiplicity of intersection with the hyperplane $\{L_1 = 0\}$.

It remains to estimate the multiplicity $\text{mult}_o\{f_1, \dots, f_{i-1} = L_1 = 0\}$. This is somewhat harder, since the form L_1 depends on the tuple (f_1, \dots, f_{i-1}) and for this reason is not a form of general position with respect to that tuple. Note that for a generic tuple (f_1, \dots, f_{i-1}) the set $\Pi(f_1, \dots, f_{i-1})$ does not depend on the choice of the form L . Therefore, the set

$$\Pi \subset \mathcal{P}_{\leq 2, M}^{i-b} \times \mathcal{P}_{2, M}^{b-1} \times \mathcal{P}_{1, M},$$

defined as the closure of the set of tuples

$$(f_1, \dots, f_{i-1}, L_1 \in \Pi(f_1, \dots, f_{i-1}))$$

for generic tuples $(f_1, \dots, f_{i-1}) \in [\bar{B}]_{i-1}$, does not depend on the choice of the form L , either. Since that form of general position L does not take part in the subsequent constructions, to simplify the notations we write L instead of L_1 , if it does not generate a confusion.

Obviously, the set Π is invariant with respect to the action of the group $GL_M(\mathbb{C})$, therefore the projection

$$\pi: \Pi \rightarrow \mathcal{P}_{1, M},$$

$$\pi: (f_1, \dots, f_{i-1}, L) \mapsto L,$$

is surjective and all its fibres are of the same dimension. Since the codimension of the closed set Π (with respect to the ambient space $\mathcal{P}_{\leq 2, M}^{i-b} \times \mathcal{P}_{2, M}^{b-1} \times \mathcal{P}_{1, M}$) does not exceed the number

$$\text{codim}[\bar{B}]_{i-1} + \gamma_i = a - (M + b - i)b,$$

for a generic linear form $L \in \mathcal{P}_{1, M}$ the codimension of the fibre $\pi^{-1}(L) \subset \mathcal{P}_{\leq 2, M}^{i-b} \times \mathcal{P}_{2, M}^{b-1}$ is bounded from above by the same number $a - (M + b - i)b$.

Now for a generic tuple $(f_1, \dots, f_{i-1}) \in \pi^{-1}(L)$ there are two options:

1) either the differentials $(df_1|_{\{L=0\}}(o), \dots, df_{i-b}|_{\{L=0\}}(o))$ remain linearly independent (an equivalent formulation: the subspace

$$\{df_1(o) = \dots = df_{i-b}(o) = 0\} \tag{2}$$

is not contained in the hyperplane $\{L = 0\}$),

2) or the rank of the set of linear forms

$$df_1(o)|_{\{L=0\}}, \dots, df_{i-b}(o)|_{\{L=0\}}$$

drops by one (an equivalent formulation: the subspace (2) is contained in the hyperplane $\{L = 0\}$).

In the case 1) set $\alpha = \alpha(B) = 1$, in the case 2) set $\alpha = \alpha(B) = 0$. Furthermore, let

$$\bar{B}_L \subset \mathcal{P}_{\leq 2, M-1}^{i-b} \times \mathcal{P}_{2, M-1}^{b-1}$$

be the closure of the set

$$\{(f_1|_{\{L=0\}}, \dots, f_{i-1}|_{\{L=0\}}) \mid (f_1, \dots, f_{i-1}) \in \pi^{-1}(L)\}.$$

Let us consider first the case 1). Here for a generic tuple $(g_1, \dots, g_{i-1}) \in \bar{B}_L$ the differentials

$$dg_1(o), \dots, dg_{i-b}(o)$$

are linearly independent, and for $j \geq i - b + 1$ we have $dg_j(o) = 0$. Now we argue as in Sec. 1.4: the set \bar{B}_L is the result of reducing to the standard form of a certain closed set $C \subset \mathcal{P}_{\leq 2, M-1}^{i-1}$. The set C is obtained from \bar{B}_L by the procedure, which is converse to the procedure of reducing to the standard form. Obviously, $\varepsilon(C) = b - 1$ and

$$\begin{aligned} \text{codim } C &= \text{codim } \bar{B}_L - (i - b)(b - 1) + (M - 1)(b - 1) = \\ &= \text{codim } \bar{B}_L + (M + b - i - 1)(b - 1), \end{aligned}$$

so that taking into account the estimate

$$\text{codim } \bar{B}_L \leq \text{codim } \pi^{-1}(L) \leq a - (M + b - i)b$$

we obtain the inequality

$$\text{codim } C \leq a - (M + b - i) - \alpha(b - 1).$$

Since

$$\text{mult}_o\{f_1 = \dots = f_{i-1} = L = 0\} = \text{mult}_o\{f_1|_{\{L=0\}} = \dots = f_{i-1}|_{\{L=0\}} = 0\},$$

we obtain the final upper estimate for that multiplicity: it can not exceed the number

$$\mu_{i-1, M-1}(a - (M + b - i) - \alpha(b - 1), e - \alpha).$$

(Recall that in the case under consideration $\alpha = 1$, and in the inequalities above the codimension is taken with respect to the natural ambient spaces, each of the sets \bar{B}_L , C , $\pi^{-1}(L)$ has its own ambient space.)

Now let us consider the case 2). Here for a generic tuple $(g_1, \dots, g_{i-1}) \in \bar{B}_L$ the rank of the system of linear functions $dg_1(o), \dots, dg_{i-b}(o)$ is equal to $i - b - 1$. We may assume that the first $i - b - 1$ of them are linearly independent, and $dg_{i-b}(o)$ is their linear combination. For $j \geq i - b + 1$ we get, as above, that $dg_j(o) = 0$. In the case 2) the set \bar{B}_L is not the result of reducing to the standard form. However, replacing g_{i-b} by the uniquely determined linear combination

$$g_{i-b}^+ = g_{i-b} - \sum_{j=1}^{i-b-1} \lambda_j g_j,$$

$dg_{i-b}^+(o) = 0$, and taking the closure, we get the set

$$\bar{C} \subset \mathcal{P}_{\leq 2, M-1}^{i-b-1} \times \mathcal{P}_{2, M-1}^b,$$

which already is the result of reducing to the standard form of a certain closed subset $C \subset \mathcal{P}_{\leq 2, M-1}^{i-1}$. Taking into account the $GL_i(\mathbb{C})$ -invariance of the original subvariety B , we conclude that all values of the coefficients λ_j in the formula for g_{i-b}^+ are realized, so that

$$\begin{aligned} \text{codim } \bar{C} &\leq \text{codim } \bar{B}_L + (i - b - 1) - (M - 1) \leq \\ &\leq a - (M + b - i)(b + 1) \end{aligned}$$

and for that reason

$$\begin{aligned} \text{codim } C &\leq a - (M + b - i)(b + 1) + (M - 1)b - (i - b - 1)b = \\ &= a - (M + b - i), \end{aligned}$$

whereas $\varepsilon(C) = b$. Since in the case under consideration $\alpha = 0$, we get that the multiplicity of the intersection

$$\text{mult}_o\{f_1 = \dots = f_{i-1} = L = 0\},$$

as in the case 1), can be estimated from above by the number

$$\mu_{i-1, M-1}(a - (M + b - i) - \alpha(b - 1), e - \alpha),$$

which completes the proof of Theorem 1. Q.E.D.

Remark 1.4. For $a \leq M$ the claim of Theorem 1 and its proof remain valid for spaces of polynomials of arbitrary degree $d \geq 2$. In the beginning of the proof of Theorem 1 (Sec. 1.4) the polynomial f_i should be taken in the form gL_1 , where g is a generic polynomial of degree $(d - 1)$ (it is sufficient to require that the differential $dg(o)$ is a linear form of general position with respect to a generic tuple (f_1, \dots, f_{i-1})), and $L_1 \in \mathcal{P}_{1, M}$ is a linear form. The proof given above works without any modifications.

§2. Asymptotic estimates

In this section, using the inductive inequality of Theorem 1, we obtain upper bounds for the numbers $\mu_{i,M}(a, b)$ and $\mu_i(a)$ and consider their asymptotics for sufficiently high values of M .

2.1. Estimates for the small values of $\varepsilon = b$. As we mentioned above, for the trivial reasons $\mu_{i,M}(a, 0) = 1$.

Example 2.1. Let us obtain an upper bound for the numbers $\mu_{i,M}(a, 1)$. We get

$$\mu_{i,M}(a, 1) \leq 1 + \mu_{i-1, M-1}(a - (M + 1 - i), 1 - \alpha_1).$$

If $\alpha_1 = 1$, then $\mu_{i,M}(a, 1) \leq 2$. If $\alpha_1 = 0$, then Theorem 1 can be applied once again. Assume that the value of the parameter α is 0 at the first k steps:

$$\alpha_1 = \dots = \alpha_k = 0.$$

Applying Theorem 1 k times, we get:

$$\begin{aligned} \mu_{i,M}(a, 1) &\leq 1 + \mu_{i-1, M-1}(a - (M + 1 - i), 1) \leq \\ &\leq 2 + \mu_{i-2, M-2}(a - 2(M + 1 - i), 1) \leq \\ &\quad \dots \\ &\leq k + \mu_{i-k, M-k}(a - k(M + 1 - i), 1). \end{aligned}$$

This is possible if the inequality

$$a \geq (k + 1)(M + 1 - i)$$

holds. Therefore, the maximal possible number k of steps, at which the parameter α keeps the value 0, is equal to

$$\left\lceil \frac{a}{M + 1 - i} \right\rceil - 1.$$

As a result, we obtain the estimate

$$\mu_{i,M}(a, 1) \leq \left\lceil \frac{a}{M + 1 - i} \right\rceil + 1,$$

in particular, $\mu_{M,M}(a, 1) \leq a + 1$. Note that the last estimate is precise: the equality $\varepsilon = 1$ means that the complete intersection

$$\{f_1 = \dots = f_{M-1} = 0\}$$

is a smooth curve at the point o . The condition of tangency of order $a \leq M$ imposes on the polynomial f_M at most a independent conditions. As a result we obtain the equality

$$\mu_{M,M}(a, 1) = a + 1.$$

Example 2.2. Let us obtain an upper bound for the numbers $\mu_{i,M}(a, 2)$. Again let us assume that at the first k steps the value of the parameter α is equal to 0. This is possible, if the inequality $a \geq (k+1)(M+2-i)$ holds. After k applications of Theorem 1 we obtain the inequality

$$\begin{aligned} \mu_{i,M}(a, 2) &\leq \sum_{j=1}^k \mu_{i-j, M-j+1}(a - j(M+2-i), 1) + \\ &\quad + \mu_{i-k, M-k}(a - k(M+2-i) - 1, 1). \end{aligned}$$

Taking the maximal possible value of k and using the estimate of the previous example, we get

$$\mu_{i,M}(a, 2) \leq \frac{1}{2} \left\lceil \frac{a}{M+2-i} \right\rceil \left(\left\lceil \frac{a}{M+2-i} \right\rceil + 1 \right) + 2.$$

For $i = M$ this estimate can be made slightly more precise:

$$\mu_{M,M}(a, 2) \leq \frac{1}{2} \left\lceil \frac{a}{2} \right\rceil \left(\left\lceil \frac{a}{2} \right\rceil + 1 \right) + \delta,$$

where $\delta = 1$, if a is even, and $\delta = 2$, if a is odd.

In a similar way one can obtain an upper estimate for $\mu_{i,M}(a, b)$ for $b = 3, 4, \dots$: applying several times Theorem 1, we can ensure that in the right hand side of the inequality the value of the parameter ε were equal to $b-1$ in all summands, after which we can apply the inequality for $\mu_{i,M}(a, b-1)$, obtained at the previous step.

2.2. The general method. Applying Theorem 1 k times in the same way as we did in Examples 2.1 and 2.2, under the assumption that the value of the parameter α is equal to 0, we obtain the inequality

$$\begin{aligned} \mu_{i,M}(a, b) &\leq \sum_{j=1}^k \mu_{i-j, M-j+1}(a - j(M+b-i), b-1) + \\ &\quad + \mu_{i-k, M-k}(a - k(M+b-i) - (b-1), b-1). \end{aligned}$$

Note that the inequality $a \geq (k+1)(M+b-i)$ holds. However, it is difficult to obtain in this way a general estimate for $\mu_{i,M}(a, b)$, reducing it to the estimate for the numbers with $\varepsilon = b-1$, because of the difficult formulas, which are hard to follow. However, we may conclude that a multiple application of Theorem 1 yields the estimate

$$\mu_{i,M}(a, b) \leq \sum_{j, N, a', b'} \mu_{j, N}(a', b') \quad (3)$$

for a certain set of tuples (j, N, a', b') (possibly, with repetitions of the same tuple), and in the end, the estimate

$$\mu_{i,M}(a, b) \leq \sum_{j, N, a'} \mu_{j, N}(a', 0), \quad (4)$$

where in the right hand side all components are equal to 1, so that it is sufficient to estimate from above the number of components, which is equal to the number of inductive steps — applications of Theorem 1. For this purpose, with each term in the right hand side of the inequality (3) we associate a word

$$\omega = \tau_1 \tau_2 \dots \tau_K$$

in the alphabet $\{A, B_0, B_1\} \ni \tau_i$, describing the “origin” of that term. With the term $\mu_{i,M}(a, b)$ itself in the tautological estimate

$$\mu_{i,M}(a, b) \leq \mu_{i,M}(a, b)$$

we associate the empty word. Let

$$\mu_{i,M}(a, b) \leq \sum_{w \in W'} \mu[w] \quad (5)$$

be the new writing of the inequality (3), where each term $\mu_{j,N}(a', b')$ in the right hand side corresponds to a word $w \in W'$ and is written as $\mu[w]$. Let us choose and fix such a term with $b' \geq 1$. According to the proof of Theorem 1, this term gives an upper estimate for the number $\mu(B')$, where $B' \subset \mathcal{P}_{\leq 2, N}^j$ a certain $GL_j(\mathbb{C})$ -invariant irreducible subvariety of codimension a' with $\varepsilon(B') = b'$. Now, applying Theorem 1, we replace (keeping the inequality) the term $\mu_{j,N}(a', b')$ by the sum of two new numbers $\mu[w_1] + \mu[w_2]$, where $\mu[w_1]$ and $\mu[w_2]$ correspond to the first and second terms in the right hand side of the inequality (1), respectively. Here $w_1 = wA$ and $w_2 = wB_\alpha$, where $\alpha = \alpha(B) \in \{0, 1\}$. This determines the procedure of constructing the words w in a unique way. It is clear that with each word at most one term in (3) is associated. Thus to obtain an upper estimate for $\mu_{i,M}(a, b)$, we need to estimate the number of words, to which terms in the inequality (4) correspond.

For instance, in Example 2.1 the set of words is

$$A, B_0A, \dots, \underbrace{B_0 \dots B_0}_k A, \underbrace{B_0 \dots B_0}_k B_1.$$

Remark 2.1. Let $\nu: \{A, B_0, B_1\} \rightarrow \{A, B\}$ be the map of the three-letter alphabet into the two-letter one, given by $\nu(A) = A$, $\nu(B_\alpha) = B$,

$$\nu: w = \tau_1 \dots \tau_K \mapsto \bar{w} = \nu(\tau_1) \dots \nu(\tau_K)$$

the corresponding map of the set of words. Then for any inequality (5), obtained by an application of Theorem 1, the restriction $\nu|_{W'}$ is injective. Indeed, each application of Theorem 1 replaces some word w by the pair of words wA and wB_α , where the value of the parameter α is uniquely determined.

Now with each summand $\mu_{j,N}(a', b')$ (or with the word w , corresponding to that summand) we associate the triple of integer-valued parameters (a', b', Δ') , where $\Delta' = N + b' - j$. By Theorem 1,

- for the word wA the associated triple is $(a' - \Delta', b' - 1, \Delta')$,
- for the word wB_0 it is the triple $(a' - \Delta', b', \Delta')$,
- for the word wB_1 it is the triple $(a' - \Delta' - (b' - 1), b' - 1, \Delta' - 1)$.

Recall now that the term $\mu_{j,N}(a', b')$ is well defined only if the inequality $a' \geq b'\Delta'$ holds.

Let W be the set of words, corresponding to the summands of the right hand side of the inequality (4). Let $W_l \subset W$ be the subset, consisting of the words, in which precisely l letters are B_1 . Obviously,

$$W = \coprod_{l=0}^b W_l$$

(the union is disjoint), so that

$$\#W = \sum_{l=0}^b \#W_l.$$

It remains to estimate from above the number of elements in each of the sets W_l .

Lemma 2.1. *The inequality*

$$\#W_l \leq \binom{A_l}{b-l}$$

holds, where $A_l = \left\lfloor \frac{a-lb}{\Delta-l} \right\rfloor$.

Proof. Consider first the case $l = 0$. In the word $w \in W_l$ there are no letters B_1 , whereas the letter A occurs precisely b times, since to the word w corresponds the triple $(a', 0, \Delta')$, and the letter B_0 does not change the value of the parameter $\varepsilon = b'$. On the other hand, since the letter B_1 does not occur, we get $\Delta' = \Delta = M + b - i$, and the inequality $a' \geq 0$ implies that the length of the word w does not exceed $A_0 = \lfloor a/\Delta \rfloor$. Thus $\#W_0$ does not exceed the number of ways of putting b letters A on at most A_0 positions. However, the last letter in the word $w \in W_0$ can be only the letter A , by the same reason that A decreases the value of $\varepsilon = b'$ by 1, and B_0 does not change it. Therefore, $\#W_0$ does not exceed the number of ways of putting b letters A on A_0 positions, which is what we need.

Now let us consider the case of an arbitrary $l \leq b$.

Lemma 2.2. *The length of a word $w \in W_l$ does not exceed A_l .*

Accepting the claim of Lemma 2.2, let us complete the proof of Lemma 2.1. Obviously, the letter A occurs in a word $w \in W_l$ precisely $(b-l)$ times. We associate with the word w the corresponding way of putting $(b-l)$ letters A on A_l positions.

We claim that this map is injective. (This immediately implies Lemma 2.1.) Indeed, assume that this is not true: there are two distinct words $w_1 \neq w_2$ in W_l

with the same distribution of the letter A . Assume that the length $|w_1|$ of the word w_1 does not exceed the length $|w_2|$. Changing to the two-letter alphabet $\{A, B\}$, we conclude that the letter w_1 is a left segment of the word w_2 and

$$w_2 = w_1 B_{\alpha_1} \dots B_{\alpha_k}$$

for some $\alpha_1, \dots, \alpha_k \in \{0, 1\}$. However, the parameter $\varepsilon = b'$ of the word w_1 is already equal to 0, which implies that $w_1 = w_2$. Q.E.D. for Lemma 2.1.

Proof of Lemma 2.2. Let us control the length $|w|$ of the word $w \in W_l$ by the decreasing of the parameter $a' \geq 0$. The slower it decreases, the longer can be the word. Assume that the letter B_1 occupies the positions

$$k_1, k_1 + k_2, \dots, k_1 + k_2 + \dots + k_l,$$

where $k_i \geq 1$. On each segment

$$[k_1 + \dots + k_j + 1, k_1 + \dots + k_{j+1} - 1]$$

of the word w (provided it is non-empty) the value of the parameter $\varepsilon = b'$ can get smaller by, at most, $k_{j+1} - 1$, whereas the value of the parameter Δ' remains the same. Therefore, to the left segment of the word w of length $k_1 + \dots + k_l$ corresponds the value

$$\begin{aligned} a' &\geq a - k_1 \Delta - (b - k_1) - \\ &\quad - k_2(\Delta - 1) - (b - k_1 - k_2) - \\ &\quad \dots \\ &\quad - k_l(\Delta - (l - 1)) - (b - k_1 - \dots - k_l) = \\ &= (a - lb) - (\Delta - l)(k_1 + \dots + k_l). \end{aligned}$$

After the position $(k_1 + \dots + k_l)$ the value of the parameter Δ' remains the same and is equal to $(\Delta - l)$. Therefore,

$$|w| \leq k_1 + \dots + k_l + \left\lceil \frac{(a - lb) - (\Delta - l)(k_1 + \dots + k_l)}{\Delta - l} \right\rceil = A_l.$$

Q.E.D. for Lemma 2.2.

Corollary 2.1. *The inequality*

$$\mu_{i,M}(a, b) \leq \sum_{l=0}^b \binom{A_l}{b-l}$$

holds, where $A_l = \left\lceil \frac{a - lb}{M + b - i - l} \right\rceil$.

2.3. An asymptotic estimate for a high dimension. Obtaining compact upper estimates for the numbers $\mu_i(a)$, which could be used for particular computations, presents a non-trivial problem. The inequality of Corollary 2.1 is too

complicated and not very visual. However, in one case it is easy to derive from it a simple and precise estimate.

Example 2.3. Assume that $M = m^2$ is a full square. Then the following equality holds:

$$\mu_{M,M}(M, m) = 2^m.$$

Indeed, all numbers $A_l = m$ are the same, so that we get

$$\mu_{M,M}(M, m) \leq \sum_{l=0}^m \binom{m}{l} = 2^m.$$

On the other hand, obviously $\mu_{M,M}(M, m) \geq 2^m$. Q.E.D.

Now let us consider the general case for $i = M$ and the maximal possible codimension $a = M$. Set $\xi(M) = \mu_M(M)$. Since

$$\xi(M) = \max_{1 \leq b \leq \lfloor \sqrt{M} \rfloor} \mu_{M,M}(M, b),$$

by Corollary 2.1 we get

$$\xi(M) \leq \sqrt{M} \max \left(\begin{array}{c} \left\lfloor \frac{M - lb}{b - l} \right\rfloor \\ b - l \end{array} \right),$$

where the maximum is taken over $b \in \{1, \dots, \lfloor \sqrt{M} \rfloor\}$ and $l \in \{1, \dots, b\}$. Now elementary computations with binomial coefficients and an application of the Stirling formula give the following result. Set

$$\omega = \max_{s \in [1, \infty)} \left[2s \ln s - \left(s - \frac{1}{s} \right) \ln(s^2 - 1) \right].$$

Proposition 2.1. *For sufficiently high M the inequality*

$$\xi(M) \leq \sqrt{M} e^{\omega \sqrt{M}}$$

holds, where e is the base of the natural logarithm.

§3. Systems of equations with the set of solutions of “incorrect” dimension

In this section, we prove Proposition 1.1.

3.1. Systems of homogeneous equations. In the space $\mathcal{P}_{d,M+1}^i$ of systems of homogeneous polynomials (p_1, \dots, p_i) of degree $d \geq 2$ in the variables z_0, \dots, z_M consider the closed subset Y , consisting of such tuples (p_1, \dots, p_i) , that the set

$$\{p_1 = \dots = p_i = 0\} \subset \mathbb{P}^M$$

is of “incorrect” codimension $\leq i - 1$.

Proposition 3.1. *The codimension of the subset Y in the space $\mathcal{P}_{d,M+1}^i$ is not less than*

$$\min_{b \in \{0, \dots, i-1\}} \{((b+1)d - b)(M - b) + 1\}.$$

Proof. It follows directly from [3, Proposition 4], taking into account that the degrees of the polynomials (p_1, \dots, p_i) are equal. Q.E.D.

Corollary 3.1. *For $i \leq M - 1$ the codimension of the subset Y in the space $\mathcal{P}_{d,M+1}^i$ is not less than $dM + 1$, and for $i = M$ it is not less than $(d - 1)M + 2$.*

Proof. Since $d \geq 2$, the quadratic function

$$\gamma(b) = b^2(1 - d) + b(dM - M - d) + dM + 1$$

of the variable b is negative definite and attains its maximum at

$$b_* = \frac{dM - M - d}{2(d - 1)} > 0.$$

Therefore, the minimum of this function on the set $\{0, \dots, i - 1\}$ is attained either for $b = 0$ (and equal to $dM + 1$), or for $b = i - 1$. It is easy to check that $\gamma(M - 1) = 2(dM - d - M) + 5 \geq dM + 1$, which proves the first claim of the corollary. Furthermore, $\gamma(M) = (d - 1)M + 2 \leq dM + 1$, which proves the second claim. Q.E.D.

3.2. Systems of non-homogeneous equations. Let us prove Proposition 1.1. In the space $\mathcal{P}_{d,M+1}^i \times \mathbb{P}^M$ consider the closed algebraic set \mathcal{Y} , consisting of such pairs $((p_1, \dots, p_i), x \in \mathbb{P}^M)$, that the corresponding set of zeros $\{p_1 = \dots = p_i = 0\}$ has an irreducible component of “incorrect” codimension $\leq i - 1$, passing through the point x . Furthermore, denote by the symbol Y_b , $b = 0, \dots, i - 1$, the closed subset in Y , consisting of such tuples (p_1, \dots, p_i) , that the codimension of the set of zeros $\{p_1 = \dots = p_i = 0\}$ does not exceed $i - 1 - b$; in particular, $Y_0 = Y$. By the methods of [3, Sec. 3] it is easy to check that $\text{codim}_Y Y_b \geq 2b$ (in fact, the estimate is much stronger). This implies that

$$\dim \mathcal{Y} = \dim Y + M - i + 1.$$

By the symbols π_1 and π_2 denote the projections of the direct product $\mathcal{P}_{d,M+1}^i \times \mathbb{P}^M$ onto the first and second factors, respectively. Obviously, $\pi_1(\mathcal{Y}) = Y$. Furthermore, $\pi_2(\mathcal{Y}) = \mathbb{P}^M$, and all the fibres $\pi_2^{-1}(x) \cap \mathcal{Y} = \mathcal{Y}_x$ are of the same dimension

$$\dim \mathcal{Y} - M = \dim Y - i + 1.$$

On the other hand, the space $\mathcal{P}_{\leq d,M}^i$ can be naturally identified with the closed subset of codimension i in $\pi_2^{-1}(x) \cong \mathcal{P}_{d,M+1}^i$, consisting of such tuples (p_1, \dots, p_i) , that

$$p_1(x) = \dots = p_i(x) = 0.$$

It is clear that \mathcal{Y}_x is contained in that subset, so that the codimension of \mathcal{Y}_x with respect to $\mathcal{P}_{\leq d,M}^i$ is equal to

$$\text{codim}(Y \subset \mathcal{P}_{d,M+1}^i) - 1.$$

Applying Corollary 3.1, we complete the proof of Proposition 1.1.

3.3. Precision of the estimates. How precise are the estimates of Proposition 1.1? The following example shows that for $i = M$ the estimate is sharp. Let $L \ni o$ be an arbitrary line passing through the origin. The condition that

$$p(z_1, \dots, z_M)|_L \equiv 0$$

imposes on a polynomial of degree d precisely d independent conditions (recall that $p(0, \dots, 0) = 0$). Therefore, requiring that

$$L \subset \{p_1 = \dots = p_M = 0\}$$

we impose on the tuple of polynomials $(p_1, \dots, p_M) \in \mathcal{P}_{\leq d,M}^M$ precisely dM independent conditions. Since there is a $(M-1)$ -dimensional family of lines, passing through the point o , the set of tuples (p_1, \dots, p_M) such that the closed set $\{p_1 = \dots = p_M = 0\}$ contains a line passing through the point o , is of codimension $(d-1)M + 1$ in the space $\mathcal{P}_{\leq d,M}^M$. Therefore, the estimate of Proposition 1.1 is sharp. In particular, the set of tuples (p_1, \dots, p_M) , vanishing on a line, forms an irreducible component of the set Y . The question, what is the codimension of other components of this set, remains an open problem.

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